JOURNAL OF APPROXIMATION THEORY 76, 219–232 (1994)

Best Rational Approximation to Markov Functions

JAN-ERIK ANDERSSON

Department of Mathematics, Chalmers University of Technology and the University of Göteborg, S-412 96 Göteborg, Sweden

Communicated by Doron S. Lubinsky

Received March 16, 1992; accepted in revised form September 29, 1992

The main result concerns rational approximations to Markov–Stieltjes functions in the dual spaces L^p/H^p, $1 \leqslant p \leqslant \infty$, on the unit circle of the complex plane. For fixed n we consider approximation by rationals whose denominators have n different zeros all with double multiplicity. In general these rationals are of order $2n$ but we show that there is a best approximating one and that this one is of order n only. This result gives a new approach to previous results by Barrett and Gončar. As an example of application we study the degree of approximation of $(1-z)^\alpha$ in BMOA and uniform norms. © 1994 Academic Press, Inc.

1. INTRODUCTION

Consider a Markov function

$$\hat{\mu}(z) = \int_a^b \frac{d\mu(t)}{z-t}, \tag{1}$$

where $d\mu$ is a positive measure on $[a, b] \subset (-1, 1)$. We assume that the support of $d\mu$ contains infinitely many points. In this report we show that if we for a given n approximate $\hat{\mu}$ on $|z| = 1$ in the norm of L^p/H^p, $1 \leqslant p \leqslant \infty$, by rational functions of the form

$$r(z) = \sum_{k=1}^{n} \left(\frac{a_k}{z-t_k} + \frac{b_k}{(z-t_k)^2} \right)$$

then the best approximating r is of type

$$r(z) = \sum_{k=1}^{n} \frac{a_k}{z-t_k}.$$

This gives a somewhat different approach, e.g., to the so-called ρ^2-results by Barrett [3, 4] and Gončar [10, 11].

0021-9045/94 $6.00

At the end we apply this to the problem of estimating the order of rational approximation in H^p and dual spaces of functions of type

$$f(z)=\int_{-1}^{1}\frac{\omega(t)\,dt}{1-zt},$$

where ω is a non-negative weight function. Special attention will be paid to uniform and BMOA-norms. In that context we in particular study the order of approximation of $(1-z)^\alpha$.

The basic ideas used are given in the papers [1, 2, 5] by the author and B. Bojanov for the case $1<p<\infty$. In the situation we have here it is however possible to give an easier, complete presentation, covering also the cases $p=1$ and $p=\infty$. But in Sections 4 and 5 the influence from Bojanov [5] is strong.

2. Notations

Some of the notations that will be used are the following.

p, q are conjugate exponents, i.e., $p^{-1}+q^{-1}=1$, with $1\leqslant p\leqslant\infty$.

D is the unit disc $|z|<1$ in the complex plane $\mathbb{C}$.

L^p is on the unit circle with the convention that L^∞ is the space of continuous functions on this circle.

H^p is the usual Hardy space on D if $1\leqslant p<\infty$ but H^∞ is the space of analytic functions on D that have continuous extensions to its closure.

$\|f\|_p$ is $((1/2\pi)\int|f(z)|^p\,|dz|)^{1/p}$ if $1\leqslant p<\infty$ and is the supremum norm if $p=\infty$. (Here and in the sequel the integration is over the unit circle unless otherwise specified.)

$\|f\|_{p*}$ is the norm of f in L^p/H^p, i.e.,

$$\|f\|_{p*}=\inf\{\|f-h\|_p : h\in H^p\}.$$

$\mathbf{z}=(z_1, ..., z_n)$ is a point in $\mathbb{C}^n$.

$B_\mathbf{z}$ is the Blaschke product $B_\mathbf{z}(z)=\prod_{k=1}^n((z-z_k)/(1-\bar{z}_kz))((z-\bar{z}_k)/(1-z_kz))$.

$\mathscr{R}(\mathbf{z})$ is the class of rational functions that vanish at infinity and have the denominator $\prod_{k=1}^n(z-z_k)(z-\bar{z}_k)$. Further notations will be introduced later.

3. The Main Result

Let us assume that $\hat{\mu}$ is a fixed Markov function as in (1) and that n is a fixed positive integer. We define first for all $\mathbf{z} \in D^n$

$$e_p(\mathbf{z}) = \inf\{ \|\hat{\mu} - r\|_{p*} : r \in \mathscr{R}(\mathbf{z})\}$$

and then

$$\varepsilon_{p,n} = \varepsilon_{p,n}(\hat{\mu}) = \inf\{e_p(\mathbf{z}) : \mathbf{z} \in D^n\}.$$

For these quantities we have the following result.

Theorem 1. *Assume that $1 \leqslant p \leqslant \infty$ and that n is positive integer. Then the following holds.*

(i) *There is a $\mathbf{t} = (t_1, ..., t_n)$ with $a < t_1 < \cdots < t_n < b$ such that*

$$e_p(\mathbf{t}) = \varepsilon_{p,n}.$$

(ii) *For this $\mathbf{t}$ there is a unique $r_0 \in \mathscr{R}(\mathbf{t})$ so that*

$$\|\hat{\mu} - r_0\|_{p*} = \varepsilon_{p,n}.$$

(iii) *This r_0 is of the form*

$$r_0(z) = \sum_{k=1}^{n} \frac{a_k}{z - t_k} \qquad \text{with all} \quad a_k \geqslant 0$$

and is consequently of order n at most.

Remark. The important feature in the theorem is that though the elements of $\mathscr{R}(\mathbf{t})$ in general have order $2n$ the best approximating one has order n at most.

The outline of the proof is as follows. Parts (i) and (iii) will be given as two separate sections (4 and 5) while part (ii) is given after the following lemma.

Lemma 1. *Suppose that $1 \leqslant p \leqslant \infty$ and $\mathbf{z} \in D^n$. Then the following is true.*

(i) $e_p(\mathbf{z}) = \sup\{ |\int_a^b g(t)\, B_{\mathbf{z}}(t)\, d\mu(t)| : g \in H^q, \|g\|_q \leqslant 1\}$.

(ii) *There exists an extremal function g realizing equality in* (i).

(iii) *This g can be assumed to be positive on $(-1, 1)$ and without zeros in D.*

Proof. In the space L^p/H^p we have the equality

$$\{h/B_{\mathbf{z}} : h \in H^p\} = \mathscr{R}(\mathbf{z}). \tag{2}$$

Since $B_{\mathbf{z}}$ is a Blaschke product, we can therefore conclude that

$$e_p(\mathbf{z}) = \|B_{\mathbf{z}}\hat{\mu}\|_{p^*} \tag{3}$$

which by duality gives

$$e_p(\mathbf{z}) = \sup\left\{\left|\frac{1}{2\pi i}\int g(z)\, B_{\mathbf{z}}(z)\, \hat{\mu}(z)\, dz\right| : g \in H^q,\ \|g\|_q \leqslant 1\right\}. \tag{4}$$

Using the definition (1) of $\hat{\mu}$ we see that (3) may also be written as

$$e_p(\mathbf{z}) = \sup\left\{\left|\int_a^b g(t)\, B_{\mathbf{z}}(t)\, d\mu(t)\right| : g \in H^q,\ \|g\|_q \leqslant 1\right\}. \tag{5}$$

That takes care of part (i) of the lemma.

The duality results that we use in this context can all be read in Duren's book [6]. In particular [6, Theorem 8.1] gives part (ii) of our lemma since $B_{\mathbf{z}}\hat{\mu}$ is continuous on the unit circle.

For part (iii) we observe that if g is an extremal function for (5) then, so is $(g(z) + \overline{g(\bar{z})})/2$. So the extremal function can be assumed to be real on $(-1, 1)$. Suppose that this g has zeros in D. Since g is real on $(-1, 1)$, these zeros are symmetric with respect to $(-1, 1)$. Let B be the Blaschke product of the zeros and be chosen so that $h = g/B$ fulfils $h(0) > 0$. Then h is positive on $(-1, 1)$. Since moreover $0 < B_{\mathbf{z}}(t) < 1$ for all $t \in (-1, 1)$, except for at most a finite number of points, and the support of $d\mu$ consists of infinitely many points, we have

$$\int_a^b g(t)\, B_{\mathbf{z}}(t)\, d\mu(t) < \int_a^b h(t)\, B_{\mathbf{z}}(t)\, d\mu(t).$$

But then g is not extremal. Hence g cannot have zeros in D and is therefore also positive $(-1, 1)$. This proves Lemma 1.

We end this section by proving (ii) in Theorem 1. Again we refer to Duren [6, Theorem 8.1]. Since $B_{\mathbf{z}}\hat{\mu}$ is continuous on $|z| = 1$, we find by (2) and (3) that for each $\mathbf{z} \in D^n$ there is a unique $r_0 \in \mathscr{R}(\mathbf{z})$ such that $\|\hat{\mu} - r_0\|_{p^*} = e_p(\mathbf{z})$.

4. The Extremal $\mathbf{t}$

We first prove that real poles suffices.

Lemma 2. $\inf\{e_p(\mathbf{z}) : \mathbf{z} \in D^n\} = \inf\{e_p(\mathbf{x}) : \mathbf{x} \in [a, b]^n\}$.

Proof. We observe that

(i) if $w = u + iv \in D$, then

$$\left|\frac{t-w}{1-\bar{w}t}\right| \geqslant \left|\frac{t-u}{1-ut}\right| \qquad \text{for} \quad t \in (-1, 1);$$

(ii) if $u \in (-1, 1)\backslash[a, b]$ then with either $c = a$ or $c = b$

$$\left|\frac{t-u}{1-ut}\right| \geqslant \left|\frac{t-c}{1-ct}\right| \qquad \text{for} \quad t \in [a, b].$$

Consequently for each $\mathbf{z} \in D^n$ there exists an $\mathbf{x} \in [a, b]^n$ such that $B_\mathbf{z}(t) \geqslant B_\mathbf{x}(t)$ for all $t \in [a, b]$. Using Lemma 1 and (5) we find that $e_p(\mathbf{x}) \leqslant e_p(\mathbf{z})$. Now we are ready to prove the existence of optimal poles.

Lemma 3. *For $1 \leqslant p \leqslant \infty$ there exists a $\mathbf{t} = (t_1, ..., t_n) \in [a, b]^n$ such that*

(i) $e_p(\mathbf{t}) = \inf\{e_p(\mathbf{z}) : \mathbf{z} \in D^n\}$

(ii) $a < t_1 < \cdots < t_n < b$.

Proof. Using Lemma 2 we see that there exists a convergent sequence $(\mathbf{x}_k)_1^\infty$ of points in $[a, b]^n$ such that

$$\lim_{k \to \infty} e_p(\mathbf{x}_k) = \inf\{e_p(\mathbf{z}) : \mathbf{z} \in D^n\}.$$

Let $\mathbf{t} = \lim \mathbf{x}_k$. Then $\mathbf{t} \in [a, b]^n$ and $B_{\mathbf{x}_k} \to B_\mathbf{t}$ uniformly on $[a, b]$. With g as an optimal function for $\mathbf{t}$ in (5) we see that

$$e_p(\mathbf{t}) = \int_a^b g(t)\, B_\mathbf{t}(t)\, d\mu(t) = \lim \int_a^b g(t)\, B_{\mathbf{x}_k}(t)\, d\mu(t) \leqslant \lim e_p(\mathbf{x}_k).$$

Hence $\mathbf{t}$ is optimal and part (i) is proved.

In order to prove (ii) we assume that the coordinates of $\mathbf{t}$ take on m different values $\tau_1 < \cdots < \tau_m$. We represent $\mathbf{t}$ as the matrix

$$\mathbf{t} = \begin{pmatrix} \tau_1 & \tau_2 \cdots \tau_m \\ \mu_1 & \mu_2 \cdots \mu_m \end{pmatrix},$$

where μ_k is the number of times the corresponding τ_k is repeated in $\mathbf{t}$. Then $\sum_{k=1}^{m} \mu_k = n$ and $\mu_k \geqslant 1$ for each k. Our assertion that $m = n$ follows if we can show that $\mu_k = 1$ for all k.

Suppose that $\mu_k > 1$ for some k. For $h > 0$ but close to 0 we form points $\mathbf{t}_h \in D^n$ by

$$\mathbf{t}_h = \begin{pmatrix} \tau_1 \cdots \tau_{k-1} & \tau_k - h & \tau_k & \tau_k + h & \tau_{k+1} \cdots \tau_m \\ \mu_1 \cdots \mu_{k-1} & 1 & \mu_k - 2 & 1 & \mu_{k+1} \cdots \mu_m \end{pmatrix}.$$

In the following estimates it is no loss of generality to assume that $\tau_k = 0$ since we may arrive at this situation after a Möbius transformation. Then

$$B_{\mathbf{t}_h}(t) = \left(\frac{t^2 - h^2}{1 - h^2 t^2}\right)^2 \cdot B(t) \qquad \text{and} \qquad B_{\mathbf{t}}(t) = t^4 B(t),$$

where

$$B(t) = t^{2\mu_k - 4} \prod_{j \neq k} \left(\frac{t - \tau_j}{1 - \tau_j t}\right)^{2\mu_j}.$$

For each h we let g_h be the extremal function in (5) with $\mathbf{z} = \mathbf{t}_h$. Since $\|g_h\|_q = 1$ we can conclude that $\{g_h\}$ is a normal family on D. So there is a sequence $(h_j)_1^\infty$ with $h_j \to 0$ such that the g_{h_j} converge uniformy on every compact subset of D to a limit function $g \in H^q$ with $\|g\|_q \leqslant 1$. Moreover,

$$\int_a^b g(t)\, B_{\mathbf{t}}(t)\, d\mu(t) = \lim_{j \to \infty} e_p(\mathbf{t}_{h_j}) \geqslant e_p(\mathbf{t})$$

so g is extremal for $\mathbf{t}$ and consequently positive on $[a, b]$.

For $e_p(\mathbf{t}_h)$ we have the inequality

$$e_p(\mathbf{t}_h) \leqslant e_p(\mathbf{t}) + \int_a^b (B_{\mathbf{t}_h}(t) - B_{\mathbf{t}}(t))\, g_h(t)\, d\mu(t).$$

Since

$$B_{\mathbf{t}_h}(t) - B_{\mathbf{t}}(t) = (-2h^2 t^2(1 - t^4) + h^4(1 - t^8))(1 - h^2 t^2)^{-2} B(t)$$

we get with $h = h_j$

$$e_p(\mathbf{t}_h) \leqslant e_p(\mathbf{t}) - h^2 \int_a^b 2t^2(1 - t^4)\, B(t)\, g(t)\, d\mu(t) + o(h^2)$$

as $j \to \infty$. Here the integral is positive so we have a contradiction to the fact that $e_p(\mathbf{t}_h) \geqslant e_p(\mathbf{t})$ for all h. Consequently we must conclude that $\mu_k = 1$

for all k. Thereby, apart from the inequalities $a < t_1$, $t_n < b$, the lemma is proved.

These remaining inequalities can be obtained in a similar way. Suppose, e.g., $t_1 = a$. Then for $h > 0$ but close to 0 we can let

$$\mathbf{t}_h = (a+h, t_2, ..., t_n)$$

and proceed as above to get a contradiction.

5. The Optimal Coefficients

As already remarked in the proof of part (ii) of the theorem in Section 3, there is for each $\mathbf{z} \in D^n$ a unique, optimal $r_0 \in \mathscr{R}(\mathbf{z})$ such that

$$\|\hat{\mu} - r_0\|_{p^*} = e_p(\mathbf{z}).$$

This r_0 can in general be represented in the form

$$r_0(z) = \sum_{k=1}^{n} \left(\frac{a_k}{z - z_k} + \frac{b_k}{(z - z_k)(z - \bar{z}_k)} \right).$$

To complete the proof of the theorem, we must prove that we have $a_k \geqslant 0$ and $b_k = 0$ for all k when $\mathbf{z} = \mathbf{t}$ as in Section 4. So let a_k and b_k be these coefficients. Fix a k and let h be a real number so close to zero that $\mathbf{z}$ with $z_j = t_j$ if $j \neq k$ and $z_k = t_k + h$, fulfils $-1 < z_1 < \cdots < z_n < 1$. Let also g_h be an extremal function in H^q with $\|g_h\|_q = 1$ so that

$$e_p(\mathbf{z}) = \int_a^b g_h(t)\, B_{\mathbf{z}}(t)\, d\mu(t).$$

We have assumed that r_0 is best in $\mathscr{R}(\mathbf{t})$, so $e_p(\mathbf{t}) = \|\hat{\mu} - r_0\|_{p^*}$. Since $\|g_h B_{\mathbf{z}}\|_q = 1$, we therefore get

$$e_p(\mathbf{t}) \geqslant \left| \frac{1}{2\pi i} \int g_h(z)\, B_{\mathbf{z}}(z) (\hat{\mu}(z) - r_0(z))\, dz \right|.$$

Using the definition of $\hat{\mu}$, we can evaluate this integral by residues. So we have

$$\begin{aligned} e_p(\mathbf{t}) &\geqslant \left| \int_a^b g_h(t)\, B_{\mathbf{z}}(t)\, d\mu(t) - \sum_{j=1}^{n} (a_j (g_h B_{\mathbf{z}})(t_j) + b_j (g_h B_{\mathbf{z}})'(t_j)) \right| \\ &= |e_p(\mathbf{z}) - a_k (g_h B_{\mathbf{z}})(t_k) - b_k (g_h B_{\mathbf{z}})'(t_k)|. \end{aligned}$$

For the last equality we have used the fact that both $g_h B_{\mathbf{z}}$ and its derivative vanish at the points t_j when $j \neq k$. All the quantities involved are real, so we can conclude that

$$e_p(\mathbf{t}) \geq e_p(\mathbf{t}) - a_k(g_h B_{\mathbf{z}})(t_k) - b_k(g_h B_{\mathbf{z}})'(t_k).$$

From this we see that

$$a_k(g_h B_{\mathbf{z}})(t_k) + b_k(g_h B_{\mathbf{z}})'(t_k) \geq 0 \tag{6}$$

for each h sufficiently close to zero.

Now $B_{\mathbf{z}}$ can be written

$$B_{\mathbf{z}}(t) = \gamma_h(t - t_k - h)^2$$

with $\{\gamma_h\}$ and $\{\gamma'_h\}$ uniformly bounded on $[a, b]$. Since $\{g_h\}$ and $\{g'_h\}$ also are uniformly bounded on $[a, b]$, we get from (6) that

$$-2g_h(t_k)\,\gamma(t_k)\,b_k h + o(h) \geq 0. \tag{7}$$

As in the previous section we can pick a sequence $(h_j)_1^\infty$ tending to $0+$ so that g_{h_j} converges uniformly on $[a, b]$ to a function g with $g(t_k) > 0$. Since moreover

$$\lim_{h \to 0} \gamma_h(t_k) = (1 - t_k^2)^{-2} \prod_{j \neq k} \left(\frac{t_k - t_j}{1 - t_k t_j}\right)^2 > 0$$

the inequality (7) gives $b_k = 0$.

The maining inequality $a_k \geq 0$ now follows directly from (6). That completes the proof of Theorem 1.

6. On the Order of Approximation

In this section we investigate the consequences of Theorem 1 for the order of rational approximation of functions

$$f(z) = \int_{-1}^{1} \frac{\omega(t)}{1 - tz}\,dt, \tag{8}$$

where $\omega(t) \geq 0$ on $(-1, 1)$.

Let R_n denote the class of rational functions of order n at most and with all its poles in $|z| > 1$. Also, let $\tilde{H}^p$ denote the class of functions h such that the function g defined by $g(z) = h(1/z)$ is in H^p and $g(0) = 0$. For each $f \in H^p$ we introduce the norm

$$\|f\|_p^* = \inf\{\|f - h\|_p : h \in \tilde{H}^p\}$$

and mearsure at first the order of approximation by

$$\rho^*_{p,n}(f) = \inf\{\|f - r\|^*_p : r \in R_n\}.$$

Later we relate this to the approximation in H^p

$$\rho_{p,n}(f) = \inf\{\|f - r\|_p : r \in R_n\}.$$

THEOREM 2. *Let $1 \leqslant p \leqslant \infty$ and let f be as in (8) where we assume that*

$$\int_{-1}^{1} \omega(t)(1 - t^2)^{-1/q}\, dt < \infty \qquad \text{if} \quad p > 1$$

and that

$$\int_{-1}^{1} \omega(t) \log(1 - t^2)^{-1}\, dt < \infty \qquad \text{if} \quad p = 1.$$

Then

$$\rho^*_{p,n}(f) \leqslant \pi \, \|\omega B_{\mathbf{z}}\|_{L^p(-1,1)} \tag{9}$$

for each $\mathbf{z} \in D^n$.

Proof. Let $0 < a < 1$ and define f_a by

$$f_a(z) = \int_{-a}^{a} \frac{\omega(t)}{1 - zt}\, dt.$$

Define the measure $d\mu$ as the restriction of $\omega(t)\, dt$ to the interval $[-a, a]$. We observe that

$$f_a(z) = \frac{1}{z}\, \hat{\mu}\left(\frac{1}{z}\right) \qquad \text{for} \quad |z| \leqslant 1.$$

To this $\hat{\mu}$ we apply Theorem 1 and get the optimal poles $\mathbf{t}$ and function $r_0 \in R_n$ so that

$$\|\hat{\mu} - r_0\|_{p^*} = e_p(\mathbf{t}).$$

For any $h \in H^p$ we have with $w = 1/z$ that

$$w\hat{\mu}(w) - wr_0(w) - wh(w) = f_a(z) - \frac{1}{z}\, r_0\left(\frac{1}{z}\right) - \frac{1}{z}\, h\left(\frac{1}{z}\right).$$

Returning to Theorem 1 we know that

$$\frac{1}{z} r_0\left(\frac{1}{z}\right) = \sum_{k=1}^{n} \frac{a_k}{1 - t_k z}.$$

With $r_1(z) = z^{-1} r_0(z^{-1})$ and $g(z) = z^{-1} h(z^{-1})$ we see that $r_1 \in R_n$ and $g \in \tilde{H}^p$. Moreover we observe that

$$\|f_a - f_1 - g\|_p = \|\hat{\mu} - r_0 - h\|_p.$$

This leads to

$$\|f_a - r_1\|_p^* \leqslant e_p(\mathbf{t})$$

and consequently since $e_p(\mathbf{t})$ is minimal also to

$$\rho_{p,n}^*(f_a) \leqslant e_p(\mathbf{z})$$

for each $\mathbf{z} \in D^n$. Using (i) in Lemma 1 and Hölder's inequality we observe that

$$e_p(\mathbf{z}) \leqslant M \, \|\omega B_{\mathbf{z}}\|_{L^p(-1,1)},$$

where

$$M = \sup\{\|g\|_{L^q(-1,1)} : g \in H^q, \|g\|_q = 1\}.$$

From the inequality by Fejér and Riesz [6, Theorem 3.13] we obtain that $M \leqslant \pi$. So we get the estimate

$$e_p(\mathbf{z}) \leqslant \pi \, \|\omega B_{\mathbf{z}}\|_{L^p(-1,1)}.$$

Since this estimate is independent of a and the conditions that we have on ω implies that $\|f - f_a\|_p \to 0$ as $a \to 1-$, we get the estimate (9) in the theorem. So the theorem is proved.

Turning to the problem of estimating $\rho_{p,n}(f)$, we know by M. Riesz' inequality [6, Theorem 4.1] that for $1 < p < \infty$ there is a constant C_p such that

$$\rho_{p,n}(f) \leqslant C_p \rho_{p,n}^*(f).$$

If $p = 1$ or $p = \infty$ we do not know whether such an inequality is valid. However, for these cases we have at least the following result.

LEMMA 4. *Assume that $p = 1$ or ∞ and that $f \in H^p$. Let $f_x(z) = f(xz)$ for $0 < x < 1$ and $|z| \leqslant 1$. Then there is a constant C such that*

$$\rho_{p,n}(f) \leqslant C \inf_{0<x<1} (\|f - f_x\|_p + \rho_{p,n}^*(f) \cdot \log(1-x)^{-1}).$$

Proof. Take functions $r \in R_n$ and $h \in \tilde{H}^p$ so that

$$\|f - r - h\|_p \leqslant 2\rho^*_{p,n}(f)$$

and observe that

$$f_x(z) - r(xz) = \frac{1}{2\pi i} \int \frac{f(\zeta) - r(\zeta) - h(\zeta)}{\zeta - xz} \, d\zeta$$

for $|z| \leqslant 1$ and $0 < x < 1$. Then the lemma follows.

In many situations the consequence of using Lemma 4 is that we get an extra logarithm in going from $\rho^*_{p,n}$ to $\rho_{p,n}$.

COROLLARY. *If there are constants C_0 and $\alpha > 0$ so that $\|f - f_x\|_p \leqslant C_0(1-x)^\alpha$ then for n sufficiently large*

$$\rho_{p,n}(f) \leqslant C\rho^*_{p,n}(f) \cdot \log \rho^*_{p,n}(f)^{-1}$$

for some constant C.

Proof. Use $x = 1 - \rho^*_{p,n}(f)^{1/\alpha}$ in Lemma 4.

Remark. The passing from $\rho^*_{\infty,n}$ to $\rho_{\infty,n}$ is of special interest since $\rho^*_{\infty,n}$ is the order of approximation in a norm that is equivalent to the BMOA-norm. For details on BMOA, see, e.g., Koosis [12].

7. APPROXIMATION OF $(1-z)^\alpha$ IN BMOA AND H^∞

As an application of the preceding results, we study the approximation of $(1-z)^\alpha$. In H^p with $1 < p < \infty$ we got a satisfactory description of the error in our paper [1]. The estimates for $\rho_{p,n}$ from below in that paper work also for the cases $p = 1$ and $p = \infty$. So here we only consider estimates from above and concentrate on the case $p = \infty$. The case $p = 1$ can be done similarly.

THEOREM 3. *Let $-1 < a < 1$ and $\alpha > 0$. Suppose that*

$$f(z) = \int_a^1 \frac{\omega(t)}{1 - zt} \, dt,$$

where $0 \leqslant \omega(t) \leqslant C_0(1-t)^\alpha$ *on* $[a, 1)$ *for some constant C_0. Then there is a constant C so that for $n = 1, 2, \ldots$ the following holds.*

(i) $\rho^*_{\infty,n}(f) \leqslant C \exp(-\pi\sqrt{2n\alpha})$

(ii) $\rho_{\infty,n}(f) \leqslant C\sqrt{n} \exp(-\pi\sqrt{2n\alpha})$.

Proof. It follows as in Ganelius [7] that the points $\mathbf{z}$ can be chosen so that

$$|\omega(t)\, B_{\mathbf{z}}(t)| \leqslant C \exp(-\pi \sqrt{2n\alpha})$$

for all $t \in [a, 1)$. Thus the theorem follows from Theorem 2 and the corollary of Lemma 4.

As was done in Andersson [1, Sect. 6], we get from the theorem also the same estimates when $f(z) = (1-z)^\alpha$.

We end this paper by also deriving a result for the degree of approximation of x^α on $[0, 1]$ in the uniform norm. So let for continuous functions f on $[0, 1]$

$$d_n(f) = \inf \|f - r\|_{L^\infty[0,1]},$$

where the infimum is over all rational functions of order n at most.

THEOREM 4. For each $\alpha > 0$ there is a constant C such that

$$d_n(x^\alpha) \leqslant C \sqrt{n} \exp(-2\pi \sqrt{\alpha n})$$

for all positive integers n.

Remark. From Ganelius [8] we know that there is an estimate from below without the extra factor $\sqrt{n}$ and also that this $\sqrt{n}$ in front of exp is superfluous also in the estimate from above if α is rational. For irrational α the estimate given there however is

$$d_n(x^\alpha) \leqslant C \exp(n^{1/4}) \cdot \exp(-2\pi \sqrt{\alpha n}).$$

So for that case our theorem gives an improvement though the factor $\sqrt{n}$ should not be there at all. Recently, Stahl [13] has announced this in a very precise result. Theorem 4 is therefore included mainly because it is an easy consequence of our other results.

In Andersson [1] we proved the corresponding result in $L^p[0, 1]$ without the extra $\sqrt{n}$ when $1 < p < \infty$.

Proof. The transition from the unit disc to the interval $[0, 1]$ can be done via Faber technique. This technique is described in, e.g., Ganelius [9, pp. 22–24].

First we note that

$$(1-x)^\alpha = f(x) + \frac{1}{2\pi i} \int_{|\zeta|=2} \frac{(1-\zeta)^\alpha}{\zeta - x}\, d\zeta,$$

where

$$f(x)=\frac{\sin \pi\alpha}{\pi}\int_1^2 \frac{(t-1)^\alpha}{x-t}\,dt.$$

Then it follows as in Andersson [1, Lemma 5] that

$$d_n(x^\alpha)\leqslant C_1\, d_n(f).$$

Using the Faber technique, we also have

$$d_n(f)\leqslant C_2\,\rho_{\infty,n}(\tilde{f}),$$

where

$$\tilde{f}(u)=\frac{1}{2\pi i}\int\frac{f\circ\psi(w)}{w-u}\,dw, \qquad |u|<1$$

and ψ is the mapping of $|w|>1$ onto the exterior of $[0, 1]$ defined by $\psi(w)=(w+1)^2/4w$. Taking $b>1$ so that $\psi(b)=2$ we get

$$\begin{aligned}\tilde{f}(u)&=\frac{\sin\pi\alpha}{\pi}\int_1^b\frac{(\psi(t)-1)^\alpha}{u-t}\,dt\\ &=-\frac{\sin\pi\alpha}{\pi}4^{-\alpha}\int_{1/b}^1\frac{t^{-\alpha-1}(1-t)^{2\alpha}}{1-ut}\,dt\end{aligned}$$

and can apply Theorem 3 to have

$$\rho_{\infty,n}(\tilde{f})\leqslant C_3\sqrt{n}\exp(-2\pi\sqrt{\alpha n})$$

and thereby the theorem.

References

1. J.-E. Andersson, Rational approximation to functions like x^α in integral norms, *Anal. Math.* **14** (1988), 11-25.
2. J.-E. Andersson and B. D. Bojanov, A note on the optimal quadrature in H^p, *Numer. Math.* **44** (1984), 301-308.
3. W. Barrett, On the convergence of sequences of rational approximations to analytic functions of a certain class, *J. Inst. Math. Appl.* **7** (1971), 308-323.
4. W. Barrett, On best rational approximations to certain functions defined as integrals, *J. Inst. Math. Appl.* **13** (1974), 107-116.
5. B. D. Bojanov, On the existence of optimal quadrature formulae for smooth functions, *Calcolo* **16** (1979), 61-70.
6. P. L. Duren, "Theory of H^p Spaces," Academic Press, New York/London, 1970.
7. T. Ganelius, Rational approximation in the complex plane and on the line, *Ann. Acad. Sci. Fenn. Ser. A I Math.* **2** (1976), 129-145.

8. T. Ganelius, Rational approximation to x^α on $[0, 1]$, *Anal. Math.* **5** (1979), 19–33.
9. T. Ganelius, Degree of rational approximation, *in* "Lectures on Approximation and Value Distribution," SMS, Les Presses de l'Université de Montréal, Montreal, 1982.
10. A. A. Gončar, On the degree of rational approximation to certain analytic functions, *Mat. Sb.* **105** (1978), 147–163. [In Russian]
11. A. A. Gončar, On the degree of rational approximation of analytic functions, *Trudy Mat. Inst. Steklov.* **166** (1984), 52–60. [In Russian]
12. P. Koosis, Introduction to H^p spaces *in* "London Math. Soc. Lecture Notes Series," Vol. 40, Cambridge Univ. Press, Cambridge, 1980.
13. H. Stahl, Best uniform rational approximation of x^α on $[0, 1]$, *Bull. Amer. Math. Soc.* **28** (1993), 116–122.